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Single-boson realizations of so(3) and so(2, 1)

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Abstract. Realizations of the Lie algebras so(3) and so(2, 1) are developed which are functions of integer powers of creation and destruction operators for a single boson, i.e. $(a^\dagger)^m$ and $(a)^m$, as well as a function of the associated single-boson number operator $N \equiv a^\dagger a$. Three types of realizations of the Lie algebra elements, together with the corresponding Casimir operators, are developed which depend on arbitrary m-periodic functions of N. A generic form is achieved in terms of devised m-quanta operators. The boson representation space is infinite-dimensional but may contain as many as two imbedded invariant subspaces. A distinction is made between irreducible representations of a Lie algebra which lead to *global* group representations and those which lead only to *local* (non-integrable) group representations. It is shown that the two non-unitary types of representations developed are equivalent to the unitary one. For $m=1$, specific choices for the m-periodic functions lead to realizations known in the literature.

1. Introduction

The use of boson operators (i.e. harmonic oscillator creation and destruction operators) to provide realizations of various Lie algebras now has quite an extensive history, and these realizations have found application in a number of physical systems. It is not the intention in this introduction to review in any detail this extensive history but just to touch on a few portions and provides references which serve as an entrée to the wider literature.

Undoubtedly, the first system for which such a realization was employed was that of the three-dimensional isotropic harmonic oscillator itself, since the position and momentum operators are directly expressed as linear combinations of the oscillator creation and destruction operators, and thus the angular momentum operators which provide a realization of the so(2) algebra become expressible as bilinear combinations of the oscillator operators.

Operators for a single boson, *linear* in one-dimensional harmonic oscillator creation and destruction operators but nonlinear in the associated number operator, were utilized by Holstein and Primakoff [1] to provide a unitary so(2) realization in their study of ferromagnetism. In a celebrated paper, Schwinger [2] utilized two boson pairs of operators, creation operators $a_1^\dagger$, $a_2^\dagger$ and associated destruction operators a_1, a_2, to devise a mapping (since called the Jordan–Schwinger mapping; see e.g. [3]) of the Lie algebra so(3). For applications to ferromagnetism, Dyson [4] and Maleev [5] constucted non-unitary mappings for su(2) that were linear in one-dimensional harmonic oscillator creation and destruction operators; subsequently it was shown [6] that this mapping could be unitarized to become the Holstein–Primakoff mapping.

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There have been many applications of boson realizations of Lie algebras in regard to nuclear physics and an extensive review of such applications as well as the underlying group theoretical analysis has recently been given by Klein and Marshalek [6], who have provided an exhaustive bibliography (581 references!). In one of the nuclear applications, $3N$ boson operators are employed [7] to describe SU(N), with relevance to the classification of states of N particles moving in a common harmonic oscillator potential.

The general group theoretical question of realizations of Lie algebras by means of boson operators has also been addressed over the years and continues to attract a fair amount of interest [8–11]. In general, the focus has been upon realizations which involve *many-boson* creation and destruction operators, or realizations which are linear or bilinear in such operators (as well as a function of the number operator) for a *single boson*.

In this paper, in the context of realizations of the Lie algebras so(3) and so(2, 1), we shall develop realizations which are functions of integer powers of creation and destruction operators for a single boson, i.e. for a given m depending on $(a^\dagger)^m$ and $(a)^m$, as well as the associated single-boson number operator. This shall be done by a very straightforward constructive procedure utilizing the Lie algebra commutation relations. The realizations obtained have a functional dependence on a number of arbitrary m-periodic functions of the single-boson number operator. It is shown that the corresponding Casimir operators associated with these realizations simply depend on one of the arbitrary m-periodic functions. In general, these realizations are infinite-dimensional, though for certain choices of the aforementioned m-periodic functions, they possess one or two imbedded finite-dimensional invariant subspaces. While the realizations, for different m, may be useful for the purpose of imbedding additional algebras in a common subspace, the realizations for different m may be put into a single form by utilization of m-quanta operators $b_m{}^\dagger$, b_m and an associated m-quanta number operator. These m-quanta operators are constructed in terms of the single-boson operators.

An analysis is given of the invariant subspaces and the associated irreducible representations obtained for the Lie algebras. A distinction is drawn between realizations of a Lie algebra, satisfying the requisite commutation relations, which may be related to a *local* representation (i.e. in the neighbourhood of the identity) but *not* a *global* representation of an associated Lie group, and a realization of a Lie algebra which may be related to a *global* representation. For realizations associated with such global group representations, generalizations of the Holstein–Primakoff [1] realization (the $m=1$ case) are obtained.

In this paper, three parallel constructive processes are used, one leading to an explicitly unitary realization for so(3), one providing constrained non-unitary realizations, and one giving non-constrained non-unitary realizations. It is shown that for a given m the constrained non-unitary as well as the non-constrained non-unitary realizations are equivalent to that of the unitary one, generalizing the procedure by which the Dyson–Maleev [4, 5] mapping becomes unitarized to give the Holstein–Primakoff mapping.

2. Group algebra

In the following, with the aid of *single*-boson operators $a^\dagger$ and a, we shall construct infinite-dimensional realizations of the Lie algebra of the special orthogonal group

SO(3) which leaves invariant the bilinear quantity $(x_1)^2+(x_2)^2+(x_3)^3$, as well as the Lie algebra of the non-compact group SO(2, 1) which leaves invariant the bilinear quantity $(x_1)^2+(x_2)^2-(x_3)^2$. As is well known (see e.g. [12]), the three elements of the associated Lie algebras, so(3) and so(2, 1) respectively, obey the commutator algebra

$$[L_3, L_\pm]=\pm L_\pm \qquad [L_+, L_-]=\lambda L_3$$

where $\lambda=+1$ for so(3) and $\lambda=-1$ for so(2, 1).

The operator that commutes with the three elements, namely the Casimir operator which characterizes the realization, is given (up to a multiplicative and additive constant) by the expression

$$\mathscr{C}=2L_-L_++\lambda L_3(L_3+1).$$

The Hermitian conjugate one-dimensional oscillator (Bose) creation and destruction operators $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$ satisfy the commutation relations

$$[\boldsymbol{a}, \boldsymbol{a}]=[\boldsymbol{a}^\dagger, \boldsymbol{a}^\dagger]=0 \qquad [\boldsymbol{a}, \boldsymbol{a}^\dagger]=1$$

so for the 'number' operator $N\equiv\boldsymbol{a}^\dagger\boldsymbol{a}$,

$$[N, \boldsymbol{a}^\dagger]=\boldsymbol{a}^\dagger \qquad [N, \boldsymbol{a}]=-\boldsymbol{a}.$$

For integer $m\geqslant 1$, an operator function $f(N)$ of N satisfies the relationships

$$(\boldsymbol{a}^\dagger)^m f(N)=f(N-m)(\boldsymbol{a}^\dagger)^m$$

$$(\boldsymbol{a})^m f(N)=fN+m)(\boldsymbol{a})^m$$

which may easily be established by considering the matrix elements of these operator equations on the basis $|\mathsf{n}\rangle$ for which

$$\mathbf{N}|\mathsf{n}\rangle=\mathsf{n}|\mathsf{n}\rangle \qquad \boldsymbol{a}^\dagger|\mathsf{n}\rangle=\sqrt{\mathsf{n}+1}|\mathsf{n}+1\rangle \qquad \boldsymbol{a}|\mathsf{n}\rangle=\sqrt{\mathsf{n}}|\mathsf{n}-1\rangle.$$

Furthermore, it follows that

$$\boldsymbol{a}^m(\boldsymbol{a}^\dagger)^m=\frac{(N+m)!}{N!} \qquad \text{and} \qquad (\boldsymbol{a}^\dagger)^m\boldsymbol{a}^m=\frac{N!}{(N-m)!}.$$

3. Realizations involving integer powers of $a^\dagger$ and a: Hermitian L_+, L_- connection

Infinite-dimensional basis realizations of the so(3) and so(2, 1) Lie algebras, involving integer powers m in the Bose creation and destruction operators $\boldsymbol{a}^\dagger$ and $\boldsymbol{a}$, may be constructed in a straightforward fashion. In this section, we identify the m-realization of the elements of the algebra(s) by the following scheme:

$$L_+\rightarrow L_+(m)=(\boldsymbol{a}^\dagger)^m f_m(N) \qquad L_-\rightarrow L_-(m)=f_m(N)(\boldsymbol{a})^m \qquad L_3\rightarrow L_3(m)=g_m(N)$$

where m is an integer $\geqslant 1$. The association ascribed here to $L_+(m)$ and $L_-(m)$ is such that they are Hermitian conjugates of one another, i.e.

$$L_-(m)^\dagger=L_+(m)$$

if $f_m(N)$ is a real function of N.

The functions $g_m(N)$, and $f_m(N)$ are required to satisfy the commutator algebra

$$[g_m(N), (\boldsymbol{a}^\dagger)^m f_m(N)]=+(\boldsymbol{a}^\dagger)^m f_m(N)$$

$$[g_m(N), f_m(N)(\boldsymbol{a})^m]=-f_m(N)(\boldsymbol{a})^m$$

$$[(\boldsymbol{a}^\dagger)^m f_m(N), f_m(N)(\boldsymbol{a})^m]=\lambda g_m(N).$$

with m being $1, 2, 3, \ldots$. This leads to the following conditions:

$$\{g_m(N) - g_m(N-m) - 1\} = 0$$

$$\frac{N!}{(N-m)!}\{f_m(N-m)\}^2 - \frac{(N+m)!}{N!}\{f_m(N)\}^2 - \lambda g_m(N) = 0.$$

The general solution for $g_m(N)$ satisfying the first of these equations is

$$g_m(N) = \frac{1}{m}(N - m\mathcal{T}_m(N))$$

where $\mathcal{T}_m(N)$ is *any* m-periodic function† of N, i.e. $\mathcal{T}_m(N) = \mathcal{T}_m(N+m)$.

To solve the second equation, the nonlinear difference equation for $f_m(N)$ is linearized by considering the quantity $[f_m(N)]^2$, which leads to the general solution

$$f_m(N) = \left\{\frac{(-\lambda)N!}{2m^2(N+m)!}\left[N + m(\tfrac{1}{2} - \mathcal{T}_m(N) - \mathcal{R}_m(N))\right]\right.$$
$$\left.\times\left[N + m(\tfrac{1}{2} - \mathcal{T}_m(N) + \mathcal{R}_m(N))\right]\right\}^{1/2}.$$

Here, the operator $\mathcal{R}_m(N)$ is an m-periodic function (possibly different than $\mathcal{T}_m(N)$).

The Casimir operator $\mathcal{C}_m$ for this m-realization may be evaluated by considering its $\langle n' | | n\rangle$ matrix elements, which gives the result

$$\mathcal{C}_m = \lambda\{-\tfrac{1}{4} + (\mathcal{R}_m(N))^2\}.$$

Thus, the characterization of the m-realization depends only on one of the m-periodic functions that occurs, namely $\mathcal{R}_m(N)$. It may be easily verified that $\mathcal{R}_m(N)$, any m-periodic function of N, satisfies the commutation relations

$$[\mathcal{R}_m(N), (a^\dagger)^m f(N)] = [\mathcal{R}_m(N), f(N)(a)^m] = \mathcal{R}_m(N), N] = 0.$$

For a given m, the commutation of this Casimir operator with the elements of the Lie algebra does not imply that these realizations are irreducible. $\mathcal{R}_m(N)$ is not necessarily a multiple of the (infinite-dimensional) unit matrix, which it would have to be by Schur's lemma if these representations were irreducible. Even if $\mathcal{R}_m(N)$ were chosen to be a constant, independent of N, so that in the infinite-dimensional $|n\rangle$ boson basis it would be proportional to a unit matrix, this would still not imply irreducibility: only if it were *known* that the realization were irreducible would it then be necessary that the matrix representing a commuting operator must be a multiple of the unit matrix.

It is observed that the operators $L_+(m)$, $L_-(m)$, and $L_3(m)$, which are functions of $(a^\dagger)^m$, a^m and N, only connect eigenstates of N whose eigenvalues are m apart. Thus, the results obtained for the Lie algebra operators may be cast in a form, using *m-quanta* creation, annihilation and number operators (defined in the following), which utilizes only those eigenstates actually connected by the Lie algebra operators.

† Any m-periodic function, say $\mathcal{T}_m(N)$, may be represented by the Fourier series expansion

$$\mathcal{T}_m(N) = \sum_{k=-\infty}^{\infty} e^{i2\pi Nk/m}\tilde{\mathcal{T}}_m(k)$$

where $\tilde{\mathcal{T}}_m(k)$ are arbitrary coefficients.

3.1. *M-realization of Lie algebras in terms of m-quanta boson operators*

Consider m-quanta states $|n_m\rangle$ which are eigenstates of the number operator $N = a^\dagger a$ with eigenvalues which are m-quanta apart:

$$|n_m\rangle \equiv |nm\rangle \qquad |n_m+1\rangle \equiv |(n+1)m\rangle \qquad |n_m-1\rangle \equiv |(n-1)m\rangle \qquad \text{etc.}$$

These m-quanta eigenstates can be considered as eigenstates of a constucted m-quanta number operator N_m such that $N_m|n_m\rangle = n_m|n_m\rangle$. The m-quanta number operator itself may be written in terms of constructed m-quanta creation and annihilation operators $b_m^\dagger$ and b_m by the usual prescription

$$N_m = b_m^\dagger b_m$$

where the effect of $b_m^\dagger$ and b_m on the eigenstates of N_m is required to be

$$b_m^\dagger|n_m\rangle = (n_m+1)^{1/2}|n_m+1\rangle$$

$$b_m|n_m\rangle = (n_m)^{1/2}|n_m-1\rangle$$

so that

$$[b_m^\dagger, b_m] = 1.$$

We construct the operators $b_m^\dagger$ and b_m by assuming they have the form

$$b_m^\dagger = (a^\dagger)^m W_m(N) \qquad b_m = W_m(N)(a)^m.$$

Satisfaction of the commutation relations then leads to a solution for $W_m(N)$. The result is

$$b_m^\dagger = (a^m)^\dagger \left[\frac{1}{m}\,\frac{N!}{(N+m-1)!}\right]^{1/2}$$

$$b_m = \left[\frac{1}{m}\,\frac{N!}{(N+m-1)!}\right]^{1/2} a^m$$

so $N_m = N/m$.

The inverse relationships between the m-quanta operators and the single quantum operators are

$$(a^\dagger)^m = b_m^\dagger \left[\frac{[m(N_m+1)]!}{(mN_m)!\,(N_m+1)}\right]^{1/2}$$

$$a^m = \left[\frac{[m(N_m+1)]!}{(mN_m)!\,(N_m+1)}\right]^{1/2} b_m$$

and $N = mN_m$.

Note, m-periodic functions of N become 1-periodic functions of N_m, i.e.

$$\mathcal{T}_m(N) = \mathcal{T}_1(N_m) \qquad \mathcal{R}_m(N) = \mathcal{R}_1(N_m).$$

In terms of the m-quanta operators, the m-realizations of the Lie algebras obtained in the previous section become simply

$$L_+(m) = (a^\dagger)^m f_m(N) = b_m^\dagger f_1(N_m)$$

$$L_-(m) = f_m(N) a^m = f_1(N_m) b_m$$

and

$$L_3(m)=g_m(N)=g_1(N_m).$$

The Casimir operator becomes simply

$$\mathscr{C}_m=\lambda\{-\tfrac{1}{4}+(\mathscr{R}_1(N_m))^2\}$$

so it is just a multiple of the unit matrix in terms of the basis of the eigenstates of N_m.

Thus, the m-realizations are *just the same* as the $m=1$ realizations in which the m-quanta creation, destruction and number operators $(\boldsymbol{b}_m^\dagger, \boldsymbol{b}_m, N_m)$ are used in place of the 1-quantum creation, destruction and number operators $(\boldsymbol{a}^\dagger, \boldsymbol{a}, N)$. The matrix elements of the Lie algebras *in terms of the basis of eigenstates* of N_m are the same for different positive integer values of m. One could consider the basis of eigenstates of N_m as an *m-contracted* basis of the eigenstates of N.

The reason why it may be desirable to use the eigenstates of N as a basis rather than the m-contracted basis is that one may wish to embed the Lie algebras of so(3) or so(2, 1) in a larger algebra whose additional elements have matrix elements between eigenstates of N associated with eigenvalues that are *not m* apart. Thus, *utilization of the m-representation* in terms of $\boldsymbol{a}^\dagger, \boldsymbol{a}, N$, rather than $\boldsymbol{b}_m^\dagger, \boldsymbol{b}_m, N_m$, *provides the flexibility to attempt such imbedding.*

3.2. Nullspaces, invariant subspaces and finite irreducible representations

In terms of the m-quanta operators, the three elements of the m-realizations of the algebras are

$$L_+(m)=\boldsymbol{b}_m^\dagger\left(\frac{-\lambda}{2(N_m+1)}\left[N_m+\frac{1}{2}-\mathscr{T}_1-\mathscr{R}_1\right]\left[N_m+\frac{1}{2}-\mathscr{T}_1+\mathscr{R}_1\right]\right)^{1/2}$$

$$L_-(m)=\left(\frac{-\lambda}{2(N_m+1)}\left[N_m+\frac{1}{2}-\mathscr{T}_1-\mathscr{R}_1\right]\left[N_m+\frac{1}{2}-\mathscr{T}_1+\mathscr{R}_1\right]\right)^{1/2}\boldsymbol{b}_m$$

and

$$L_3(m)=N_m-\mathscr{T}_1$$

where, in terms of the eigenstate basis of N_m, the quantities $\mathscr{T}_1(N_m)$ and $\mathscr{R}_1(N_m)$ are simply numbers $\mathscr{T}_1$ and $\mathscr{R}_1$ independent of N_m. The Casimir operator also just becomes the number

$$\mathscr{C}_m=\lambda(\mathscr{R}_1+\tfrac{1}{2})(\mathscr{R}_1-\tfrac{1}{2}).$$

We note that the preceding expressions are even in $\mathscr{R}_1$.

We will now systematically look at the conditions on $\mathscr{T}_1$ and $\mathscr{R}_1$ which yield nullspaces of $L_-(m)$ and $L_+(m)$, i.e. those states $|n_m^-\rangle$, $|n_m^+\rangle$ which satisfy $L_-(m)|n_m^-\rangle=0$, $L_+(m)|n_m^+\rangle=0$.

First of all, it is obvious that $|0\rangle$ is always a $|n_m^-\rangle$ nullspace regardless of the values of $\mathscr{T}_1$ and $\mathscr{R}_1$. The conditions concerning the existence of other nullspaces may be put into three categories described in the following:

(*A*) *No nullspace states of* $L_\pm(m)$. Conditions: $\mathscr{T}_1+\mathscr{R}_1\neq j+\frac{1}{2}$, $\mathscr{T}_1-\mathscr{R}_1\neq j'+\frac{1}{2}$ where j and j' are non-negative integers. In this situation there are no nullspace states of $L_+(m)$, and $L_-(m)$ has only the nullspace state $|0\rangle$. Thus there are *no* invariant *finite* subspaces embedded in the infinite-dimensional m-quanta boson space, which would provide finite-dimensional realizations of the algebras. (The infinite-dimensional realizations in the infinite-dimensional m-quanta basis of states will be discussed later.)

(*B*) *Only one nullspace state of* $L_\pm(m)$. Conditions: $\mathcal{T}_1 = j + \frac{1}{2} \pm \mathfrak{R}_1$ (two possibilities), where j is a non-negative integer. For either of these situations there is an m-quanta nullspace state $|j\rangle$ of $L_+(m)$, and furthermore the next 'higher' state is a nullspace state of $L_-(m)$, i.e.

$$L_+(m)|j\rangle = 0 \qquad \text{and} \qquad L_-(m)|j+1\rangle = 0.$$

Thus, there is one $(j+1)$-dimensional invariant *finite* subspace imbedded in the infinite-dimensional m-quanta boson space. This yields a $(j+1)$-dimensional irreducible finite representation of the algebra(s). For these two possibilities for $\mathcal{T}_1$, the operator L_3 becomes

$$L_3(m) = (N_m - j - \tfrac{1}{2} \mp \mathfrak{R}_1).$$

Thus, for general $\mathfrak{R}_1$, we see that although a $(j+1)$-dimensional finite represenation of the so(3) *algebra* is produced, this does *not* provide the global generators of the SO(3) group, since global generators only arise if $L_3(m)$ acting on the basis states produce an integer value. (For the generators of SU(2), whose algebra is isomorphic to SO(3), 'half-integer' values are also allowed.) Consequently, only for 'half-integer' or integer choices for $\mathfrak{R}_1$ do these finite representations of the algebra correspond to the global generators of the relevant groups.

Moreover, for the algebras, there is also the remaining invariant infinite-dimensional subspace. Using the notation { } to separate the invariant subspaces of the m-quanta realizations, we may depict these subspaces in the following fashion:

$$\{|0\rangle, |1\rangle, \ldots, |j\rangle\}, \{|j+1\rangle, \ldots, \rightarrow |\infty\rangle\}.$$

(*C*) *Two nullspace states of* $L_\pm(m)$. Conditions: Simultaneously $\mathcal{T}_1 + \mathfrak{R}_1 = j + \frac{1}{2}$ and $\mathcal{T}_1 - \mathfrak{R}_1 = j' + \frac{1}{2}$, where j and j' are non-negative integers. To facilitate the discussion, we shall distinguish two subcases:

(i) Relationship $j = j'$. This is a degenerate case in which the two nullspace states of $L_+(m)$ actually collapse into a single state. The invariant subspace structure is then

$$\{|0\rangle, |1\rangle, \ldots, |j\rangle\}, \{|j+1\rangle, \ldots, \rightarrow |\infty\rangle\}$$

but in contradistinction to category (*B*) above, the Casimir number now has no arbitrary $\mathfrak{R}_1$ dependence but has the value $\mathscr{C}_m = -\lambda/4$.

The action of $L_3(m)$ on the nullspace states of the $(j+1)$-dimensional irreducible finite subspace is

$$L_3(m)|0\rangle = -(j + \tfrac{1}{2})|0\rangle \qquad L_3(m)|j\rangle = -(\tfrac{1}{2})|j\rangle.$$

Thus, for $\lambda = 1$, this $(j+1)$-finite-dimensional representation of the so(3) algebra does indeed represent the global spinor generators of the SU(2) group, since $L_3(m)$ acting on the basis states produces half-integer values. The correspondence with the usual basis and eigenvalues for $L_3(m)$ can be made by relating the eigenvalues n_m to shifted values k by the relation $k = n_m + (j+1)/2$, and defining basis states $\|\,\rangle\rangle$ by the relation $|n_m\rangle = \|n_m - j/2\rangle\rangle$. Then, for the finite $(j+1)$-dimensional representation, the shifted eigenvalues of $L_3(m)$ range from $-k/2$ to $k/2$ *as* $\|\,\rangle\rangle$ goes from $\|-j/2\rangle\rangle$ to $\|j/2\rangle\rangle$. This recovers the Holstein–Primakoff [1] representation for $m = 1$, i.e. when one-quantum states are used.

where $\mathcal{T}_m(N)$ is *any* m-periodic function of N.

To solve the second equation, the nonliner difference equation for $f_m(N)$ is linearized by considering the difference equation for the product $f_m(N)f_m(N-m)$, which leads to a general solution that provides the product difference equation

$$f_m(N)f_m(N-m) = -\frac{\lambda(N-m)!}{2(m)^2N!}\{N-m(\tfrac{1}{2}+\mathcal{T}_m(N)+\mathcal{R}_m(N))\} \times\{N-m(\tfrac{1}{2}+\mathcal{T}_m(N)-\mathcal{R}_m(N))\}$$

where the operator $\mathcal{R}_m(N)$ is an m-periodic function (possibly different than $\mathcal{T}_m(N)$.

Now, we use the fact [14] that the gamma function ratio

$$S_m(N;\rho_m) \equiv \frac{\sqrt{2m}\,\Gamma[(2m)^{-1}(N+m+m\rho_m)]}{\Gamma[(2m)^{-1}(N+m\rho_m)]}$$

satisfies the relation

$$S_m(N;\rho_m)S_m(N-m;\rho_m) = (N-m+m\rho_m)$$

if $\rho_\mu(N)$ is any m-periodic function of N. Thus, a particular solution, $f_m^{(p)}(N)$ of the nonlinear difference equation for $f_m(N)$, is given by

$$f_m^{(p)}(N) = \frac{(-\lambda/2)^{1/2}}{m}\,\frac{S_m(N;\tfrac{1}{2}-\mathcal{T}_m+\mathcal{R}_m)S_m(N;\tfrac{1}{2}-\mathcal{T}_m-\mathcal{R}_m)}{\prod_{k=0}^{m-1} S_m(N;1-k/m)}.$$

To obtain the *general* solution of the product difference equation, we note that the ratio of the general solution $f_m(N)$ to the particular solution $f_m^{(p)}(N)$,

$$r_m(N) = f_m(N)/f_m^{(p)}(N)$$

satisfies the relation

$$r_m(N)r_m(N-1) = 1.$$

Thus, it follows that

$$e^{iN\pi/m}\ln[r_m(N)] = \mathcal{Q}_m(N)$$

where $\mathcal{Q}_m(N)$ is *any* m-periodic function of N, i.e. $\mathcal{Q}_m(N+m) = \mathcal{Q}_m(N)$. Thus, it follows that the *general* solution for $f_m(N)$ is

$$f_m(N) = f_m^{(p)}(N)\exp\{[\cos(N\pi/m) - i\sin(N\pi/m)]\mathcal{Q}_m(N)\}$$

where $\mathcal{Q}_m$ is any m-periodic function of N. Note that the solution of the original postulated commutation relations involves three arbitrary m-periodic functions: $\mathcal{T}_m(N)$, $\mathcal{R}_m(N)$ and $\mathcal{Q}_m(N)$.

The Casimir operator $\mathcal{C}_m$, evaluated by considering $\langle n'||n\rangle$ matrix elements, turns out to have the same form as in section 3, namely

$$\mathcal{C}_m = \lambda\{-\tfrac{1}{4}+(\mathcal{R}_m(N))^2\}.$$

Thus, the characterization of these realizations also depends only on one of the m-periodic functions that occurs, namely $\mathcal{R}_m(N)$.

4.1. Special case in which $f_m(N)$ is independent of N

Since $S_m(N;\rho_m)$ cannot be independent of N, the only possibility for $f_m(N)$ to be *finite* subspaces embedded in the infinite-dimensional m-quanta boson space, which would provide finite-dimensional realizations of the algebras. (The infinite-dimensional realizations in the infinite-dimensional m-quanta basis of states will be discussed later.)

(*B*) *Only one nullspace state of* $L_{\pm}(m)$. Conditions: $\mathcal{T}_1 = j + \frac{1}{2} \pm \mathcal{R}_1$ (two possibilities), where j is a non-negative integer. For either of these situations there is an m-quanta nullspace state $|j\rangle$ of $L_+(m)$, and furthermore the next 'higher' state is a nullspace state of $L_-(m)$, i.e.

$$L_+(m)|j\rangle = 0 \qquad \text{and} \qquad L_-(m)|j+1\rangle = 0.$$

Thus, there is one $(j+1)$-dimensional invariant *finite* subspace imbedded in the infinite-dimensional m-quanta boson space. This yields a $(j+1)$-dimensional irreducible finite representation of the algebra(s). For these two possibilities for $\mathcal{T}_1$, the operator L_3 becomes

$$L_3(m) = (N_m - j - \tfrac{1}{2} \mp \mathcal{R}_1).$$

Thus, for general $\mathcal{R}_1$, we see that although a $(j+1)$-dimensional finite represenation of the so(3) *algebra* is produced, this does *not* provide the global generators of the SO(3) group, since global generators only arise if $L_3(m)$ acting on the basis states produce an integer value. (For the generators of SU(2), whose algebra is isomorphic to SO(3), 'half-integer' values are also allowed.) Consequently, only for 'half-integer' or integer choices for $\mathcal{R}_1$ do these finite representations of the algebra correspond to the global generators of the relevant groups.

Moreover, for the algebras, there is also the remaining invariant infinite-dimensional subspace. Using the notation { } to separate the invariant subspaces of the m-quanta realizations, we may depict these subspaces in the following fashion:

$$\{|0\rangle, |1\rangle, \ldots, |j\rangle\}, \{|j+1\rangle, \ldots, \rightarrow |\infty\rangle\}.$$

(*C*) *Two nullspace states of* $L_{\pm}(m)$. Conditions: Simultaneously $\mathcal{T}_1 + \mathcal{R}_1 = j + \frac{1}{2}$ and $\mathcal{T}_1 - \mathcal{R}_1 = j' + \frac{1}{2}$, where j and j' are non-negative integers. To facilitate the discussion, we shall distinguish two subcases:

(i) Relationship $j = j'$. This is a degenerate case in which the two nullspace states of $L_+(m)$ actually collapse into a single state. The invariant subspace structure is then

$$\{|0\rangle, |1\rangle, \ldots, |j\rangle\}, \{|j+1\rangle, \ldots, \rightarrow |\infty\rangle\}$$

but in contradistinction to category (*B*) above, the Casimir number now has no arbitrary $\mathcal{R}_1$ dependence but has the value $\mathcal{C}_m = -\lambda/4$.

The action of $L_3(m)$ on the nullspace states of the $(j+1)$-dimensional irreducible finite subspace is

$$L_3(m)|0\rangle = -(j + \tfrac{1}{2})|0\rangle \qquad L_3(m)|j\rangle = -(\tfrac{1}{2})|j\rangle.$$

Thus, for $\lambda = 1$, this $(j+1)$-finite-dimensional representation of the so(3) algebra does indeed represent the global spinor generators of the SU(2) group, since $L_3(m)$ acting on the basis states produces half-integer values. The correspondence with the usual basis and eigenvalues for $L_3(m)$ can be made by relating the eigenvalues n_m to shifted values k by the relation $k = n_m + (j+1)/2$, and defining basis states $\|\rangle\rangle$ by the relation $|n_m\rangle = \|n_m - j/2\rangle\rangle$. Then, for the finite $(j+1)$-dimensional representation, the shifted eigenvalues of $L_3(m)$ range from $-k/2$ to $k/2$ *as* $\|\rangle\rangle$ goes from $\|-j/2\rangle\rangle$ to $\|j/2\rangle\rangle$. This recovers the Holstein–Primakoff [1] representation for $m = 1$, i.e. when one-quantum states are used.

(ii) Relationship $j'=j+k$, where k is an integer $\geqslant 1$. Note: since now $2\mathscr{T}_1=j+j'+1$, and $2|\mathscr{R}_1|=|j-j'|$, we may without loss of generality take $j'>j$. For this situation $|j\rangle$ and $|j+k\rangle$ are both nullspace states of $L_+(m)$, and $|j+1\rangle$ and $|j+k+1\rangle$ are both nullspace states of $L_-(m)$. The invariant subspace structure is then

$$\{|0\rangle, |1\rangle, \ldots, |j\rangle\}, \{|j+1\rangle, \ldots, |j+k\rangle\}, \{|j+k+1\rangle, \ldots, \rightarrow|\infty\rangle\}.$$

and the Casimir number has the value

$$\mathscr{C}_m=\lambda\left(\frac{k+1}{2}\right)\left(\frac{k-1}{2}\right).$$

We see that there is one invariant *finite* subspace of dimension $j+1$, and another finite subspace of dimension k, both of which are imbedded in the infinite-dimensional m-quanta boson space. This yields an irreducible finite representation of the algebra(s) of dimension $j+1$ as well as an irreducible finite representation of dimensionality k. In addition, there is the remaining infinite-dimensional representation.

The action of $L_3(m)$ on the nullspace states of the imbedded $(j+1)$-dimensional irreducible finite subspace is

$$L_3(m)|0\rangle=-[j+(k+1)/2]|0\rangle \qquad L_3(m)|j\rangle=-[(k+1)/2]|j\rangle$$

so for $\lambda=1$, the $(j+1)$-finite-dimensional irreducible representation of the so(3) algebra does indeed give global spinor generators of the SU(2) group, since $L_3(m)$ acting on the basis states produces half-integer values. Definition of 'shifted' eigenvalues for n_m and redefined basis states $\|\rangle\rangle$, as discussed earlier, can bring the generators and basis states into correspondence with the more customary representations. Thus, once again, a generalization of the Holstein–Prinmakoff representation is obtained.

The action of $L_3(m)$ on the other nullspace states, that of the imbedded k-dimensional irreducible finite subspace, is

$$L_3(m)|j+1\rangle=-[(k-1)/2]|0\rangle \qquad L_3(m)|j+k\rangle=+[(k-1)/2]|j+k\rangle.$$

Thus, for k being an odd integer, this k-dimensional irreducible represenation of the so(3) algebra in the subspace gives the global generators of the SO(3) group as well as the integer-valued representation of the global generators of SU(2), while for k being an even integer the k-dimensional representation of the so(3) algebra in the subspace gives the global spinor generators SU(2). Again, generation of 'shifted' n_m eigenvalues and basis states $\|\rangle\rangle$ may be made to provide in this imbedded subspace a generalization of the Holstein–Primakoff representation.

3.3. *The infinite-dimensional representations of the Lie algebra*

It is known [13] that for a compact Lie group all irreducible matrix represenations are *finite*-dimensional, and particularly for SO(3) that the generator L_3 acting on basis states for which it only has diagonal matrix elements yields integral values ('half-integral' as well as integral for the group SU(2)). Thus, for $\lambda=1$, the realizations that we have constructed which do not meet these criteria *cannot* be representations of the generators associated with this group(s). Thus, one must distinguish between operators which satisfy the commutation relations of a given Lie algebra but which do *not* represent generators for an associated Lie group, and operators (satisfying the Lie algebra) which can represent the generators of an associated Lie group. In the former case, while the Lie group for infinitesimal parameters associated with the Lie algebra elements can be integrated to give a *local* (i.e. non-integrable) representation of an associated Lie group in the neighbourhood of the identity, it does not give a global representation (which requires L_3 to have integral and bounded eigenvalues for SO(3)).

3.4. *Special case $m=2$*

Returning back to the $a^\dagger$, a, N description (instead of the $b_m{}^\dagger$, b_m, N_m m-quanta description) for $L_+(m)$ and $L_-(m)$, one can inquire whether it is possible that the elements of the algebra(s) are independent of N. It is readily found by inspecting the form of $f_m(N)$ that this can happen only when $m=2$, since only then does $(N+m)!/N!$ become a quadratic denominator function in N. We can make $f_2(N)$ independent of N by *choosing* $\mathcal{R}_2(N)=\mathcal{T}_2(N)=-\frac{1}{4}$. Then $f_2(N)=(-\lambda/4)^{1/2}$ and

$$L_+(2)=(-\lambda/4)^{1/2}a^\dagger a^\dagger \qquad L_-(2)=(-\lambda/4)^{1/2}aa \qquad L_3(2)=\tfrac{1}{2}(N+\tfrac{1}{2}).$$

4. Realizations involving integer powers of $a^\dagger$ and a: a constrained non-Hermitian L_+, L_- connection

Infinite-dimensional basis realizations of the so(3) and so(2, 1) Lie algebras, containing integer power m in the Bose creation and destruction operators $a^\dagger$ and a, may also be constucted in a straightforward fashion using a different association for L_+ and L_- in terms of two functions† $f_m(N)$ and $g_m(N)$. In this section, we identify the m-realization of the elements of the Lie algebra(s) by the following scheme:

$$L_+(m)=f_m(N)(a^\dagger)^m \qquad L_-(m)=f_m(N)(a)^m \qquad L_3(m)=g_m(N)$$

where m is an integer $\geqslant 1$. Note: here $L_-(m)^\dagger$ is not $L_+(m)$ for real functions $f_m(N)$. Indeed,

$$\{f_m(N)(a)^m\}^\dagger=f_m(N-m)(a^\dagger)^m.$$

Since $L_+(m)$ and $L_-(m)$ utilize the *same* functions $f_m(N)$, we call this a *constrained* non-Hermitian realization.

Satisfaction of the Lie algebra commutation relations leads to the condition

$$\{f_m(N)\{g_m(N)-g_m(N-m)-1\}=0$$

$$\{f_m(N)f_m(N-m)\}\frac{N!}{(N-m)!}-\{f_m(N)f_m(N+m)\}\frac{(N+m)!}{N!}-\lambda g_m(N)=0.$$

The first of these equations leads to a general solution for $g_m(N)$ which is exactly of the same form as that obtained for $g_m(N)$ in section 3, namely

$$g_m(N)=\frac{1}{m}(N-m\mathcal{T}_m(N))$$

† In sections 3, 4 and 5 the notation $L_\pm(m)$, $L_3(m)$, $f_m(N)$ and $g_m(N)$ is employed for the relevant operators and associated two functions whose solution is sought. However, these are *not* the same operators/functions in these different sections. From one section to another, these differ in possible assignment of associated m-periodic functions, and in the case of $f_m(N)$ are distinct in form of solution. The same notation is used in these three sections to avoid proliferation of symbols or awkward subscript/superscipt tagging such as $f_m^{(\mathrm{III})}(N)$, $f_m^{(\mathrm{IV})}(N)$, etc. When comparisons are made of different functions in different sections, the superscript tags (using roman numerals to identify the sections) will be made explicit.

where $\mathcal{T}_m(N)$ is *any* m-periodic function of N.

To solve the second equation, the nonliner difference equation for $f_m(N)$ is linearized by considering the difference equation for the product $f_m(N)f_m(N-m)$, which leads to a general solution that provides the product difference equation

$$f_m(N)f_m(N-m) = -\frac{\lambda(N-m)!}{2(m)^2N!}\{N-m(\tfrac{1}{2}+\mathcal{T}_m(N)+\mathcal{R}_m(N))\}$$
$$\times\{N-m(\tfrac{1}{2}+\mathcal{T}_m(N)-\mathcal{R}_m(N))\}$$

where the operator $\mathcal{R}_m(N)$ is an m-periodic function (possibly different than $\mathcal{T}_m(N)$.

Now, we use the fact [14] that the gamma function ratio

$$S_m(N;\rho_m) \equiv \frac{\sqrt{2m}\,\Gamma[(2m)^{-1}(N+m+m\rho_m)]}{\Gamma[(2m)^{-1}(N+m\rho_m)]}$$

satisfies the relation

$$S_m(N;\rho_m)S_m(N-m;\rho_m) = (N-m+m\rho_m)$$

if $\rho_\mu(N)$ is any m-periodic function of N. Thus, a particular solution, $f_m^{(p)}(N)$ of the nonlinear difference equation for $f_m(N)$, is given by

$$f_m^{(p)}(N) = \frac{(-\lambda/2)^{1/2}}{m}\,\frac{S_m(N;\tfrac{1}{2}-\mathcal{T}_m+\mathcal{R}_m)S_m(N;\tfrac{1}{2}-\mathcal{T}_m-\mathcal{R}_m)}{\prod_{k=0}^{m-1}S_m(N;1-k/m)}.$$

To obtain the *general* solution of the product difference equation, we note that the ratio of the general solution $f_m(N)$ to the particular solution $f_m^{(p)}(N)$,

$$r_m(N) = f_m(N)/f_m^{(p)}(N)$$

satisfies the relation

$$r_m(N)r_m(N-1) = 1.$$

Thus, it follows that

$$e^{iN\pi/m}\ln[r_m(N)] = \mathcal{Q}_m(N)$$

where $\mathcal{Q}_m(N)$ is *any* m-periodic function of N, i.e. $\mathcal{Q}_m(N+m) = \mathcal{Q}_m(N)$. Thus, it follows that the *general* solution for $f_m(N)$ is

$$f_m(N) = f_m^{(p)}(N)\exp\{[\cos(N\pi/m) - i\sin(N\pi/m)]\mathcal{Q}_m(N)\}$$

where $\mathcal{Q}_m$ is any m-periodic function of N. Note that the solution of the original postulated commutation relations involves three arbitrary m-periodic functions: $\mathcal{T}_m(N)$, $\mathcal{R}_m(N)$ and $\mathcal{Q}_m(N)$.

The Casimir operator $\mathcal{C}_m$, evaluated by considering $\langle n'||n\rangle$ matrix elements, turns out to have the same form as in section 3, namely

$$\mathcal{C}_m = \lambda\{-\tfrac{1}{4}+(\mathcal{R}_m(N))^2\}.$$

Thus, the characterization of these realizations also depends only on one of the m-periodic functions that occurs, namely $\mathcal{R}_m(N)$.

4.1. *Special case in which $f_m(N)$ is independent of N*

Since $S_m(N;\rho_m)$ cannot be independent of N, the only possibility for $f_m(N)$ to be actually independent of N is when $m=2$; in that case $f_m^{(p)}(N)$ has two factors of S_m in its denominator which could cancel the two factors of S_m in its numerator. Explicitly, for $m=2$, $f_2(N)$ turns out to be

$$f_2(N)=\frac{(-\lambda/2)^{1/2}}{2}\,\frac{S_2(N;\tfrac{1}{2}-\mathcal{T}_2+\mathcal{R}_2)S_2(N;\tfrac{1}{2}-\mathcal{T}_2-\mathcal{R}_2)}{S_2(N;1)S_2(N;\tfrac{1}{2})}\exp\{[\cos(N\pi/2)-i\sin(N\pi/2)]\mathcal{Q}_2\}$$

where $\mathcal{T}_2$, $\mathcal{R}_2$ and $\mathcal{Q}_2$ are arbitrary 2-periodic functions of N. Now, if one *chooses* $\mathcal{Q}_2=0$, $\mathcal{T}_2=-\frac{1}{4}$, and $\mathcal{R}_2=\frac{1}{4}$, then $f_2(N)=\sqrt{-\lambda}(2)^{-3/2}$, which is indeed independent of N, and $g_2(N)=(N+\frac{1}{2})/2$. Thus, this special case retrieves the familiar commutator algebra, bilinear in the Bose operators (the same as for the special case $m=2$ discussed in section 3).

$$[(N+\tfrac{1}{2}),a^\dagger a^\dagger]=2a^\dagger a^\dagger$$
$$[(N+\tfrac{1}{2}),aa]=-2aa$$
$$[a^\dagger a^\dagger,aa]=-4(N+\tfrac{1}{2}).$$

4.2. *Equivalence of infinite-dimensional realizations*

It might be conjectured that for a given m, the constrained m-realization,

$$L_+(m)=f_m(N)(a^\dagger)^m \qquad L_-(m)=f_m(N)(a)^m \qquad L_3(m)=g_m(N)$$

(found here in section 4) is equivalent to the Hermitian m-realization (found in section 3). We will now show that these are indeed equivalent if $\mathcal{T}_m^{\mathrm{IV}}(N)$ is chosen equal to $\mathcal{T}_m^{\mathrm{III}}(N)$, and $\mathcal{R}_m^{\mathrm{IV}}(N)$ is chosen equal to $\mathcal{R}_m^{\mathrm{III}}(N)$.

Suppose it were true that, for the same m, the realization in section 4 is equivalent to a realization in section 3. Then, there must exist an operator, call it B_m, such that

$$B_m g_m^{\mathrm{IV}}(N)B_m^{-1}=g_m^{\mathrm{III}}(N)$$
$$B_m f_m^{\mathrm{IV}}(N)(a^\dagger)^m B_m^{-1}=(a^\dagger)^m f_m^{\mathrm{III}}(N)$$
$$B_m f_m^{\mathrm{IV}}(N)(a)^m B_m^{-1}=f_m^{\mathrm{III}}(N)(a)^m.$$

The first of these relationships is obviously satisfied if B_m is only a function of N. Thus, we assume that $B_m=B_m(N)$, from which it follows that $B_m^{-1}=\{B_m(N)\}^{-1}$. The matrix elements of $B_m(N)$, in terms of the eigenvector basis of N, can then be found by requiring that the second and third of the above relationships are satisfied. The result is

$$B_m(N)=\{f_m^{\mathrm{IV}}(N)\}^{-1/2}.$$

It may be easily verified that the second and third of the aforementioned relationships are satisfied since

$$[f_m^{\mathrm{IV}}(N)f_m^{\mathrm{IV}}(N-m)]^{1/2}=f_m^{\mathrm{III}}(N-m).$$

Thus, appropriate choices of the arbitrary m-periodic functions make the realizations in section 4 equivalent to those in section 3 for the same m.

5. Realizations involving integer powers of a^+ and a: arbitrarily assignable function for non-Hermitian L_+, L_-

Using *three* functions $f_m(N)$, $h(N)$ and $g_m(N)$, whose properties in this section are not necessarily related to those of previous sections, we identify the m-realization of the elements of the Lie algebra by the following scheme:

$$L_+(m)=(a^\dagger)^m f_m(N) \qquad L_-(m)=h_m(N)(a)^m \qquad L_3(m)=g_m(N)$$

where m is an integer $\geqslant 1$. Note: here $L_-(m)^\dagger$ is not $L_+(m)$ for real functions $f_m(N)$. Moreover, since $L_+(m)$ and $L_-(m)$ utilize *different* functions $f_m(N)$ and $h_m(N)$, we call this a *non-constrained* non-Hermitian realization.

Satisfaction of the Lie algebra commutation relations leads to the conditions

$$\{g_m(N)-g_m(N-m)-1\}=0$$

$$\frac{N!}{(N-m)!}\{f_m(N-m)h_m(N-m)\}-\frac{(N+m)!}{N!}\{f_m(N)h_m(N)\}-\lambda g_m(N)=0.$$

Once again, the first of these equations leads to a general solution for $g_m(N)$ which is exactly of the same form as that obtained for $g_m(N)$ in section 3, namely

$$g_m(N)=\frac{1}{m}(N-m\mathcal{T}_m(N))$$

where $\mathcal{T}_m(N)$ is *any* m-periodic function of N.

To solve the second equation, the linear difference equation for $f_m(N)$ is simplified by considering the product $f_m(N)h_m(N)$, which leads to the general solution

$$f_m(N)=-\frac{\lambda(N!)}{2(m)^2(N+m)!h_m(N)}\left\{N+m\left(\frac{1}{2}-\mathcal{T}_m(N)-\mathcal{R}_m(N)\right)\right\}$$
$$\times\left\{N+m\left(\frac{1}{2}-\mathcal{T}_m(N)+\mathcal{R}_m(N)\right)\right\}$$

where the operator $\mathcal{R}_m(N)$ is an m-periodic function.

One of the two functions $f_m(N)$, $h(N)$ may be chosen *arbitrarily*† and then the other of these functions is determined from the above relationship involving the arbitrary m-periodic functions $\mathcal{T}_m(N)$ and $\mathcal{R}_m(N)$. Evaluating the Casimir operator $\mathcal{C}_m$, we once again find it has the form

$$\mathcal{C}_m=\lambda\{-\tfrac{1}{4}+(\mathcal{R}_m(N))^2\}.$$

5.1. *Equivalence considerations*

It may be conjectured that, for a given m, the non-constrained m-realization obtained here in section 5 is equivalent to the Hermitian m-realization found in section 3) if $\mathcal{T}_m^{\mathrm{V}}(N)$ is chosen equal to $\mathcal{T}_m^{\mathrm{III}}(N)$, and $\mathcal{R}_m^{\mathrm{V}}(N)$ is chosen equal to $\mathcal{R}_m^{\mathrm{III}}(N)$.

If this is the case, then there must exist an operator $\mathcal{B}_m$ such that

$$\mathcal{B}_m g_m^{\mathrm{V}}(N)\mathcal{B}_m^{-1}=g_m^{\mathrm{III}}(N)$$

$$\mathcal{B}_m(a^\dagger)^m f_m^{\mathrm{V}}(N)\mathcal{B}_m^{-1}=(a^\dagger)^m f_m^{\mathrm{III}}(N)$$

$$\mathcal{B}_m h_m(N)(a)^m\mathcal{B}_m^{-1}=f_m^{\mathrm{III}}(N)(a)^m.$$

† This freedom of choice, for the realizations linear in the boson creation and destruction operators (i.e. for $m=1$), was discussed by Klein and Marshalek see [6], p 385) in their discussion of the Dyson–Maleev mapping.

Again, the first of these relationships is obviously satisfied if $\mathcal{B}_m$ is only a function of N. Thus, we assume that $\mathcal{B}_m = \mathcal{B}_m(N)$, from which it follows that $\mathcal{B}_m^{-1} = \{\mathcal{B}_m(N)\}^{-1}$. The matrix elements of $\mathcal{B}_m(N)$, in terms of the eigenvector basis of N, can be found by requiring that the second and third of the above relationships are satisfied. This gives the result that

$$\mathcal{B}_m(N+m) = \left(\frac{h_m(N)}{f_m^{\mathrm{V}}(N)}\right)^{1/2} \mathcal{B}_m(N).$$

This is just a recursive relationship for the matrix elements m apart of the diagonal matrix $\mathcal{B}_m(N)$. It may be easily verified that the second and third of the aforementioned relationships are satisfied since

$$[f_m^{\mathrm{V}}(N) h_m(N)]^{1/2} = f_m^{\mathrm{III}}(N).$$

Thus, with appropriate choices of the arbitrary m-periodic functions the realizations here in section 5 are equivalent to those in section 3 for the same m.

This freedom of choice was made by Klein and Marshalek ([6], p 385) in obtaining the Dyson–Maleev mapping and showing it may be unitarized to become the Holstein–Primakoff mapping. Their discussion corresponds to the $m = 1$ case with all m-periodic functions of N taken as zero.

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